Derivation of Time Difference Equation using Continuous Expression in Nonlinear Long Wave Equation and Evaluation of its Discretization Error

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We are working on the improvement of the finite difference formula of the nonlinear long-wave equation to improve the accuracy of the tsunami numerical calculation, especially its decay process. In the tsunami field, the method of [Imamura et al., 2006] is widely used for the finite difference formulas of the nonlinear long wave equation. Fluid simulations using nonlinear long-wave equations are performed in various fields, such as river engineering, oceanography and hydrology. In each field, the same method is used in many cases as in the tsunami field. However, the existing method has weak points that tsunami decay is weak, and the cause has not been elucidated yet. Therefore, we focused on the friction term of the nonlinear long-wave equation, which greatly affects the tsunami decay. At [Minami, 2019, JpGU], we showed that the accuracy of the tsunami decay process can be improved by improving the difference equation of friction term. In this process, it was shown in Fig.1 that the friction term can be solved analytically by the time difference. It was shown that the accuracy of the decay process of increased. In this paper, this form is called Simple-Implicit, following [Imamura et al., 2006]. In addition, Fig.2-2, the method widely used in tsunami numerical calculation, is called Combined-Implicit. Note that Fig.2-3 is Explicit method. Furthermore, at [Minami, 2019, the 9th Meeting of the Huge Tsunami Research Group], we compared analytical solution, Simple-Implicit, and Combined-Implicit for the process in which the gravity term and advection term did not work and the overall flow gradually weakened only with the bottom friction. We have shown that Simple-Implicit agrees with the analytic solution though it is discrete (Fig.3) and, Combined-Implicit also showed that there is a numerical oscillation reported in [Imamura et al., 2006].
Next, these computational stability in the local inertial equation has already been evaluated in previous studies [Tanaka et al., 2017], suggesting that Simple-Implicit is more stable.
Therefore, in this paper, we evaluated time discretization error between Simple-Implicit and Combined-Implicit about under various conditions. For this purpose, we have created a new tsunami calculation code. It is a 1D nonlinear long-wave equation that can calculate Simple-Implicit and Combined-Implicit at the same time, and can handle virtual terrain and virtual initial waveform. We did not perform run-up calculations to eliminate errors due to run-up calculation conditions, and using radiation boundary condition by characteristic curve.
In order to evaluate the errors of these methods, a value that serves as a criterion is necessary, but no analytical solution has yet been found for the nonlinear long-wave equation. Also, it cannot be evaluated with observation values because it contains various other errors. For that reason, we use a characteristic that regardless of the time difference method, if the integration time *dt* is shortened as much as possible, each method will produce the same result. We determined that *dt* was short enough if the value was below a certain criterion, and then set the waveform as a tentative true value. Then, we evaluated the discretization errors of the time difference by comparing with the calculation results at *dt* which is much larger than its tentative value. *dt* used in the calculation is usually in the range used in the tsunami numerical calculation. In this case, we compared *dt* with 1/128 from the original *dt*.
In this comparison, there are time discretization errors of the gravity term and the advection term, and these errors are often larger than the time discretization error of the friction term. Therefore, it was

impossible to compare the results of Simple-Implicit and Combined-Implicit. For that reason, the calculation was performed with the original dt and dt reduced to 1/128, excluding the friction term, and by taking the difference between them, the time discretization error of the advection term and the gravity term was estimated. By subtracting them from the calculation results of Simple-Implicit and Combined-Implicit, only the differences between Simple-Implicit and Combined-Implicit were extracted. The result is Fig.4. In all the calculated cases, it was shown that the error of Simple-Implicit was always small.

In summary, when solved analytical, the form of Simple-Implicit is obtained, and Simple-Implicit has better computational stability than Combined-Implicit and has less discretization error. In other words, this paper showed that Simple-Implicit is a more suitable calculation method for real-time simulations performed for disaster prevention.

Keywords: Tsunami numerical simulation, Nonlinear long wave equation, Finite difference formula

Existing difference equation of friction term is

$$gn^2\left(\frac{D_{i,j}^k+D_{i+1,j}^k}{2}\right)^{-7/3}\frac{M_{i+1/2,j}^{k+1/2}+M_{i+1/2,j}^{k-1/2}}{2}\sqrt{(M_{i+1/2,j}^{k-1/2})^2+(N_{i+1/2,j}^{k-1/2})^2}$$

For convenience, putting

$$\mathrm{M}'=\mathrm{M}_k-(\text{gravity term})-(\text{advection term})$$

$$\mathrm{fric}=dt*g*n^2*D^{-\frac{7}{3}}*\sqrt{\mathrm{M}_k{}^2+N_k{}^2}$$

we obtained

$$\mathrm{M}_{k+1}=\mathrm{M}'-\mathrm{fric}*\frac{(\mathrm{M}_k+\mathrm{M}_{k+1})}{2}$$

Next, when putting $\mathrm{N}=\alpha\cdot\mathrm{M}$, we have $\mathrm{fric}'=g*n^2*D^{-\frac{7}{3}}\cdot\sqrt{1+\alpha^2}$

The partial differential equation due to friction term is

$$\frac{\partial M}{\partial t}=-fric'*M^2$$

using separation of variables

$$\int_{\mathrm{M}(k)}^{M(k+dt)}\frac{\partial M}{M^2}=-fric'*\int_k^{k+dt}\partial t$$

$$\frac{1}{M(k)}-\frac{1}{M(k+dt)}=-fric'*dt$$

$$M(k+dt)=\frac{M(k)}{1+\boxed{fric'*dt*M(k)}}$$

$$M(k+dt)=\frac{M(k)}{1+fric}$$

namely, $$M_{k+1}=\frac{M'}{1+fric}\rightarrow M_{k+1}=\mathrm{M}'-\mathrm{fric}*M_{k+1}$$

Fig.1 Analytically solve the friction term in time difference

$$\mathrm{M}_{k+1}=\mathrm{M}'-\mathrm{fric}*\mathrm{M}_{k+1}$$ 2-1.Simple-Implicit

$$\mathrm{M}_{k+1}=\mathrm{M}'-\mathrm{fric}*\frac{(\mathrm{M}_k+\mathrm{M}_{k+1})}{2}$$ 2-2.Combined-Implicit

$$\mathrm{M}_{k+1}=\mathrm{M}'-\mathrm{fric}*\mathrm{M}_k$$ 2-3.Explicit

Fig.2 equations of Simple-Implicit, Combined-Implicit, Explicit

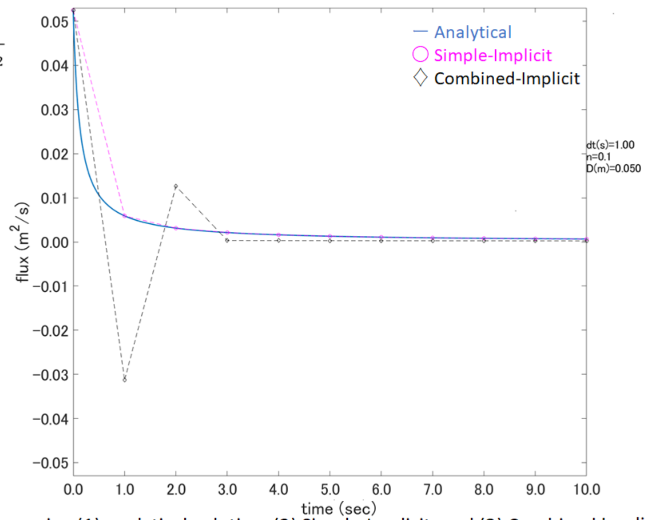

Fig.3 comparing (1) analytical solution, (2) Simple-Implicit, and (3) Combined-Implicit

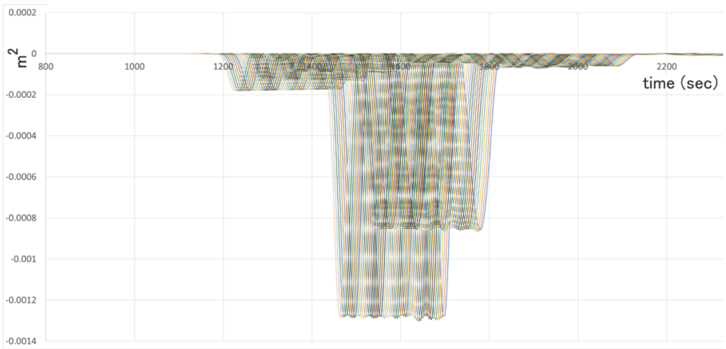

Fig.4 Timelines of discretization error of (Simple-Implicit)^2 - (Combined-Implicit)^2