幾何的性質を有する群の要素の構成に関する限界

Limits on constructions of group elements with geometric properties

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In physics [Reck et al.(1994)] and related fields such as quantum computation [Boykin et al.(1999)], control theory and mathematics [Lowenthal (1972)], issues of realization of some objects such as unitary or orthogonal operators have been discussed. In such discussions, geometric aspects are sometimes emphasized [Agrachev and Sachkov (2004)]. In this work, constructions of mathematically modeled objects, with some geometric properties, are discussed.

Suppose a group G and an element g of G is given, and we want to express g as a product of elements of G under some restrictions. For example, in [Hamada (2014)], the author has considered such a problem for $G = \mathrm{SU}(2)$ and $G = \mathrm{SO}(3)$, where the goal is to decompose g into a product of the minimum number of rotations under the restriction that only two given axes (roughly speaking, Hamiltonians) for constituent rotations are allowed.

This work presents a generic geometric inequality that shows a limit of this kind of (or similar) constructions for a wide class of objects including $\mathrm{SO}(N)$ and $\mathrm{SU}(N)$. It can be seen that the resulting bounds are tight in some non-trivial cases.

Theorem 1 *Let $\mathcal{A}$ be a set, $\mathcal{M}$ be a set of mappings from $\mathcal{A}$ to $\mathcal{A}$, and $d : \mathcal{A} \times \mathcal{A} \to \mathbb{R}$ be a mapping satisfying*

$$d(a, c) \le d(a, b) + d(b, c)$$

for any $a, b, c \in \mathcal{A}$. Suppose also that

$$d(a, b) = d(f(a), f(b))$$

for any $a, b \in \mathcal{A}$ and $f \in \mathcal{M}$. Then, for any integer ν greater than 1, any $T_1, \dots, T_\nu \in \mathcal{A}$, and any $f_1, \dots, f_\nu \in \mathcal{M}$, if

$$f_j(T_j) = T_j \quad \textit{for all } j = 1, \dots, \nu,$$

the following inequality holds:

$$d(f_\nu \circ f_{\nu-1} \circ \cdots \circ f_1(T_1), T_\nu) \le \sum_{j=1}^{\nu-1} d(T_j, T_{j+1}).$$

Note the operation of composition $\circ$ is associative. In typical applications of this theorem, $\mathcal{M}$ would be specified as an action of a group G on $\mathcal{A}$. This theorem can be shown by induction on ν.

As an example, we consider the following case, adopting the notation in [Hamada (2014)]: $\mathcal{A} = S^2$, $\mathcal{M} = \{\hat{R}_{\hat{v}}(\theta)|_{S^2} \mid \hat{v} \in S^2, \theta \in \mathbb{R}\}$, where $\hat{R}_{\hat{v}}(\theta) \in \mathrm{SO}(3)$ denotes the rotation about $\hat{v}$ by angle θ (regarded as a map rather than a matrix when the restriction $|_{S^2}$ is applied), and d is the geodesic metric on S^2. Then, we have an inequality that generalizes the two inequalities (5.2) and (5.4) in Lemma 5.1 of [Hamada (2014)]. Those two inequalities were used to obtain tight lower bounds of the minimum number of constituent rotations. The generalized inequality is presented for $\nu = 3$, the case $\nu > 3$ being obvious: For any $\hat{m}_1, \hat{m}_2, \hat{m}_3 \in S^2$ and $\theta_1, \theta_2, \theta_3 \in \mathbb{R}$,

$$\begin{aligned} d(\hat{R}_{\hat{m}_3}(\theta_3)\hat{R}_{\hat{m}_2}(\theta_2)\hat{R}_{\hat{m}_1}(\theta_1)\hat{m}_1, \hat{m}_3) \\ \le d(\hat{m}_1, \hat{m}_2) + d(\hat{m}_2, \hat{m}_3). \end{aligned} \tag{1}$$

This simple case with $\nu = 3$ is already powerful. In fact, the above inequality (i.e., the theorem) directly implies the following known result of Segercrantz, 1966 (or Davenport, 1973): Suppose that any element g in SO(3) can be written as $g = \hat{R}_{\hat{n}}(\alpha)\hat{R}_{\hat{l}}(\beta)\hat{R}_{\hat{m}}(\gamma)$ with some $\alpha, \beta, \gamma \in \mathbb{R}$. Then, $\hat{l}$ is perpendicular to both $\hat{n}$ and $\hat{m}$. [The converse is also true.] Note without loss of generality, we may assume $d(\hat{m}_j, \hat{m}_{j+1}) \in [0, \pi/2]$ for $j = 1, \dots, \nu - 1$ (if not, flip the signs of vectors).

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References.

[Agrachev and Sachkov (2004)] Agrachev and Sachkov, Control Theory from the Geometric Viewpoint, Springer, Berlin.

[Hamada (2014)] M. Hamada, Royal Society Open Science, vol. 1, 140145, `http://dx.doi.org/10.1098/rsos.140145`.

The other references can be found in [Hamada (2014)].