Application of Meta-Modeling Theory to Thin Curved Beam Using Curvilinear Coordinate System and Perturbation Expansion

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This paper presents rigorous formulation of a thin curved beam based on the meta-modeling theory proposed by the authors. A curvilinear coordinate system inherent to the configuration of the beam is introduced; it is shown that the basis vectors of the system must change on the cross section unlike the conventional treatment of the curved beam. The perturbation expansion that uses the ratio of the beam thickness and radius is applied to displacement functions. The governing equations for displacement functions are derived from a Lagrangian of continuum mechanics. Discussions are made on the limitation of the present formulation.

1. INTRODUCTION

The authors developed a meta-modeling theory[1)] to link continuum mechanics to structural mechanics. This paper presents the application of the meta-modeling theory to a thin curved beam and derives the governing equation for the dis-placement function.

The practical usefulness of the present paper is minimum, except for the case of developing a non-linear curved beam element of a finite element method. Educational significance can be large because differential geometry is learned by following the flow of the formulation.

2. Geometry of thin curved beam

Let V and N be the domain and neutral axis of a thin curbed beam; see Fig. 1. A point along N is given as $\boldsymbol{X}(S)$, and without the loss of generality, we assume $|\boldsymbol{X}'| = 1$. We define a set of unit vectors,

$$\{\boldsymbol{B}_1, \boldsymbol{B}_2, \boldsymbol{B}_3\} = \{\boldsymbol{X}', \frac{\boldsymbol{X}''}{|\boldsymbol{X}''|}, \boldsymbol{X}' \times \frac{\boldsymbol{X}''}{|\boldsymbol{X}''|}\}, \quad \text{along } N. \quad (1)$$

By definition, $\boldsymbol{B}_i$'s are mutually orthogonal, and they are genially used in analyzing the curved beam.

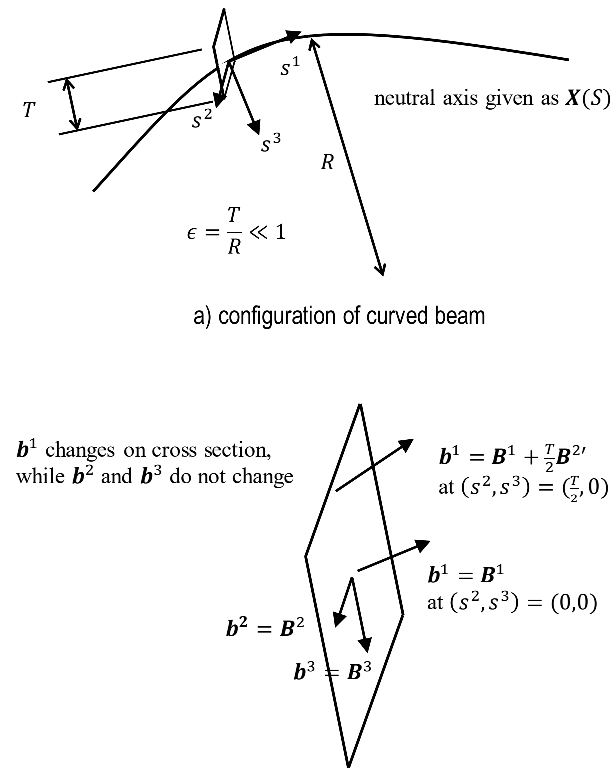
Using $\boldsymbol{B}_i$, we define s^i and $\boldsymbol{b}_i$ for $i = 1, 2, 3$ as

$$\{\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3\} = \{\boldsymbol{B}_1 + s^p \boldsymbol{B}'_p, \boldsymbol{B}_2, \boldsymbol{B}_3\} \quad \text{in } V; \quad (2)$$

s^1 is constant on a cross section normal to N with $s^1 = S$ along N; and s^2 and s^3 are naturally defined on the cross section with $s^2 = s^3 = 0$ along N. Since $\boldsymbol{b}_i$'s of Eq. (2) are regarded as a function of s^i's, the following equation holds:

$$\frac{\partial \boldsymbol{b}_j}{\partial s^i} = \frac{\partial \boldsymbol{b}_i}{\partial s^j} \quad \text{in } V. \quad (3)$$

Thus, s^i and $\boldsymbol{b}_i$ form a curvilinear coordinate system. We cannot form coordinate system using $\boldsymbol{B}_i$'s of Eq. (1), because we cannot find associte coordinates that satisfy Eq. (3) for them.



b) basis vectors on cross section

Fig.1 Configuration and basis vectors of thin curved beam.

Denoting by T and R the thickness and radius of the beam, we define

$$\varepsilon = \frac{T}{R} (\ll 1). \quad (4)$$

We can express $\boldsymbol{b}_1$ as $\boldsymbol{b}_1 = \boldsymbol{B}_1 + \varepsilon_i \boldsymbol{B}_i$ with $\varepsilon_i = s^p(\boldsymbol{B}'_p \cdot \boldsymbol{B}_i)$. This ε_i satisfies $\varepsilon_i = O(\varepsilon)$.

We can approximately compute the metric and Christoffel symbols, dropping terms of $O(\varepsilon^2)$. We can compute the gradient of a vector function, $\boldsymbol{f} = f^i \boldsymbol{b}_i$, as

$$\boldsymbol{f} \approx ((1-2\varepsilon_1)f^1_{,1} - \varepsilon_p f^1_{,p})\boldsymbol{b}_1 \otimes \boldsymbol{b}_1 \cdots,$$

where $(.)_,i = \frac{\partial ()}{\partial s^i}$.

We can compute the Jacobian, $J = \det\left[\frac{\partial x_j}{\partial s_i}\right]$ as

$$J \approx 1 + \varepsilon_1. \tag{5}$$

3. Derivation of governing equation of thin curved beam displacement

According to the meta-modeling theory[1], we use the following functional for displacement and stress functions, $\boldsymbol{u}$ and $\boldsymbol{\sigma}$:

$$\mathcal{L}[\boldsymbol{u},\boldsymbol{\sigma}] = \int_V \frac{1}{2}\boldsymbol{\sigma} : \boldsymbol{c}^{-1} : \boldsymbol{c} - \nabla\boldsymbol{u} : \boldsymbol{\sigma}\,\mathrm{d}v, \tag{6}$$

where $\boldsymbol{c}$ is elasticity tensor and $\boldsymbol{c}^{-1}$ is its inverse tensor. We use $\boldsymbol{B}_i$ or $\boldsymbol{b}_i$, respectively, to describe or compute $\boldsymbol{u}$ and $\boldsymbol{\sigma}$.

We employ the following $\boldsymbol{u}$ and $\boldsymbol{\sigma}$:

$$\begin{aligned}\boldsymbol{u} &= (U - s^p W^{p\prime} + \varepsilon_1 U)\boldsymbol{B}_1 \\ &\quad +(W^p + \varepsilon^p(U - s^q W^{q\prime})\boldsymbol{B}_p, \\ \boldsymbol{\sigma} &= \sigma\boldsymbol{B}_1 \otimes \boldsymbol{B}_1,\end{aligned} \tag{7}$$

where U and W^p are a function of s^1 only. Converting $\boldsymbol{B}_i$'s to $\boldsymbol{b}_i$'s and assuming $U,\,W \ll T$, we compute $\nabla\boldsymbol{u}$ to arrive at

$$\begin{aligned}\boldsymbol{\epsilon} &\approx ((1-\varepsilon_1)(U' - s^p W^{p\prime}) + \varepsilon_1 s^p W^{p\prime\prime} \\ &\quad +\varepsilon^p W^{p\prime})\boldsymbol{B}_1 \otimes \boldsymbol{B}_1,\end{aligned} \tag{8}$$

where $\boldsymbol{\epsilon}$ is strain, the symmetric part of $\nabla\boldsymbol{u}$.

Substituting Eqs. (7) and (8) into Eq. (6) and using the curvilinear coordinates of $J \approx 1+\varepsilon_1$, we can compute $\mathcal{L}$ as

$$\begin{aligned}\mathcal{L} &= \int_{V_s}(\frac{1}{2E}\sigma^2 - \sigma(((1-\varepsilon_1)(U' - s^p W^{p\prime\prime}) \\ &\quad +\varepsilon_1 s^p W^{p\prime} + \varepsilon^p W^{p\prime}))(1+\varepsilon_1)\,\mathrm{d}v_s.\end{aligned}$$

It follows from $\delta\mathcal{L} = 0$ with respect to σ that

$$\begin{aligned}\sigma &= E(((1-\varepsilon_1)(U' - s^p W^{p\prime}) + \varepsilon_1 s^p W^{p\prime\prime} \\ &\quad +\varepsilon^p W^{p\prime}).\end{aligned} \tag{9}$$

Substituting Eq. (9) into Eq. (6) and noting $\int s^p\,\mathrm{d}s^2\mathrm{d}s^3 = 0$, we can compute $\mathcal{L}$ as

$$\begin{aligned}\mathcal{L} &= \int(-\frac{E}{2}\sum_{p,q} A(U')^2 + I^p(W^{p\prime\prime})^2 \\ &\quad +4\kappa_{p1} I^p W^{p\prime\prime} - 2\kappa_{qp} I^q W^{p\prime} W^{q\prime\prime})\,\mathrm{d}s^1,\end{aligned} \tag{10}$$

where $\kappa_{pi} = \boldsymbol{B}'_p \cdot \boldsymbol{B}_i$, $A = \int \mathrm{d},\mathrm{d}s^2\mathrm{d}s^3$ and $I^p = \int (s^p)^2\mathrm{d},\mathrm{d}s^2\mathrm{d}s^3$. Solving $\delta\mathcal{L} = 0$ with respect to U and W^p, we finally obtain

$$\begin{aligned}&(EAU')' + 2(\kappa_p 1 E I^p W^{p\prime})' = 0, \\ &(EI^p W^{p\prime\prime} + 2(\kappa_{p1} E I^p U')'' - (\kappa_{pq} E I^p W^{q\prime})'' \\ &\quad +(\kappa_{qp} E I^q W^{q\prime\prime})' = 0.\end{aligned} \tag{11}$$

Boundary conditions are derived from $\delta\mathcal{L} = 0$, as well.

4. Discussion on formulation

The formulation of Eq. (11) takes advantage of the perturbation expansion with respect to *varepsilon* defined as Eq. (4). Usually, the ratio of T to the beam span is assumed to be small. ε is another parameter to describe the configuration of the curbed beam.

The perturbation expansion is applied to $\boldsymbol{u}$ and $\boldsymbol{\sigma}$ in addition to the curvilinear coordinate system. As *varepsilon* goes to 0, the coordinate system tends to be a Cartesian coordinate, and $\boldsymbol{u}$ and $\boldsymbol{\sigma}$ tend to be those of a straight beam. We must include terms of $O(\varepsilon)$ in $\boldsymbol{u}$ and $\boldsymbol{\sigma}$, to drop terms of $O(\varepsilon^2)$. The necessity of them is not recognized unless the perturbation expansion is applied to the curvilinear coordinate system.

As explained in Sct. 1, the formulation is educative to learn the application of differential geometry for a curvilinear coordinate system. We must emphasize that the system is defined rigorously and that the defined system is used for the computation. It is necessary to use $\boldsymbol{B}_i$'s of Eq. (1) to describe $\boldsymbol{u}$ and $\boldsymbol{\sigma}$ as usual, but they are not used to compute $\nabla\boldsymbol{u}$ unlike $\boldsymbol{b}_i$'s of Eq. (2).

The starting point of introducing the curvilinear coordinate system is that a point along N is given as a function of S. The presence of such $\boldsymbol{X}(S)$ is generally taken for granted in differential geometry. However, it is not an easy task to compute S and $\boldsymbol{X}$ when a curved beam is given; a point along N is determined using a certain Cartesian coordinate system. We need to develop a smart method of determining S and $\boldsymbol{X}$.

5. Concluding remarks

This paper applies the meta-modeling theory to a thin curved beam and presents the formulation of the governing equations for displacement functions. No physical assumptions are made in the derivation. Perturbation expansion related to the relative thickness is applied to approximately treat the beam geometry and the displacement and stress functions.

REFERENCES

1) Hori, M., L. Wijerathene, T. Ichimura and S. Tanaka: Meta-modeling for constructing model consistent with continuum mechanics, J. JSCE, 2.,pp. 269–275, 2014.

(Received empt